

Why Quantum Natural Gradients Fail

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Background

Classical VS Quantum Computer

Bit (Classical) → 0 or 1

Qubit (Quantum) → Superposition of 0 and 1 (state on a curved space)

Advantage → Massive computational possibilities

Challenge → Hard to optimize + highly noise-sensitive

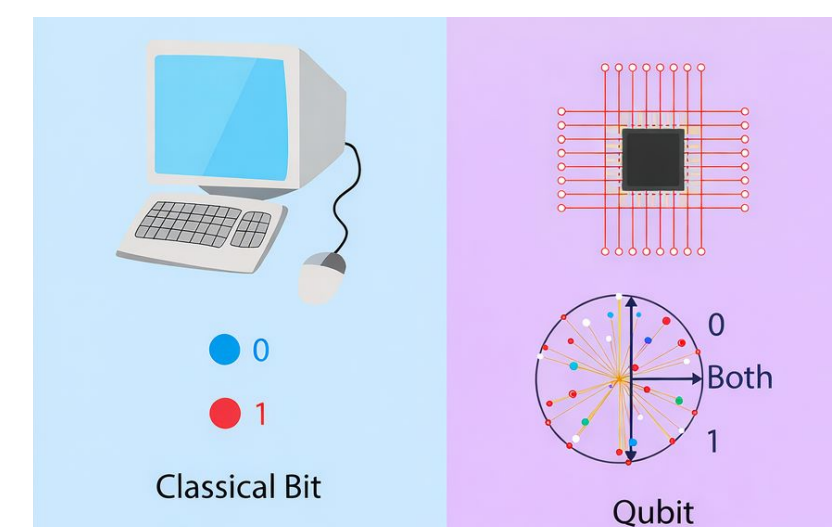
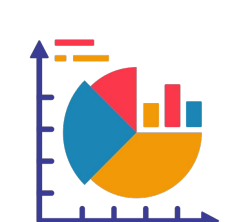
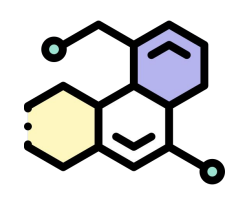
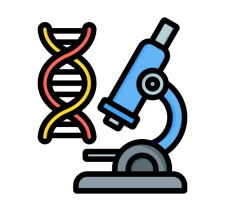
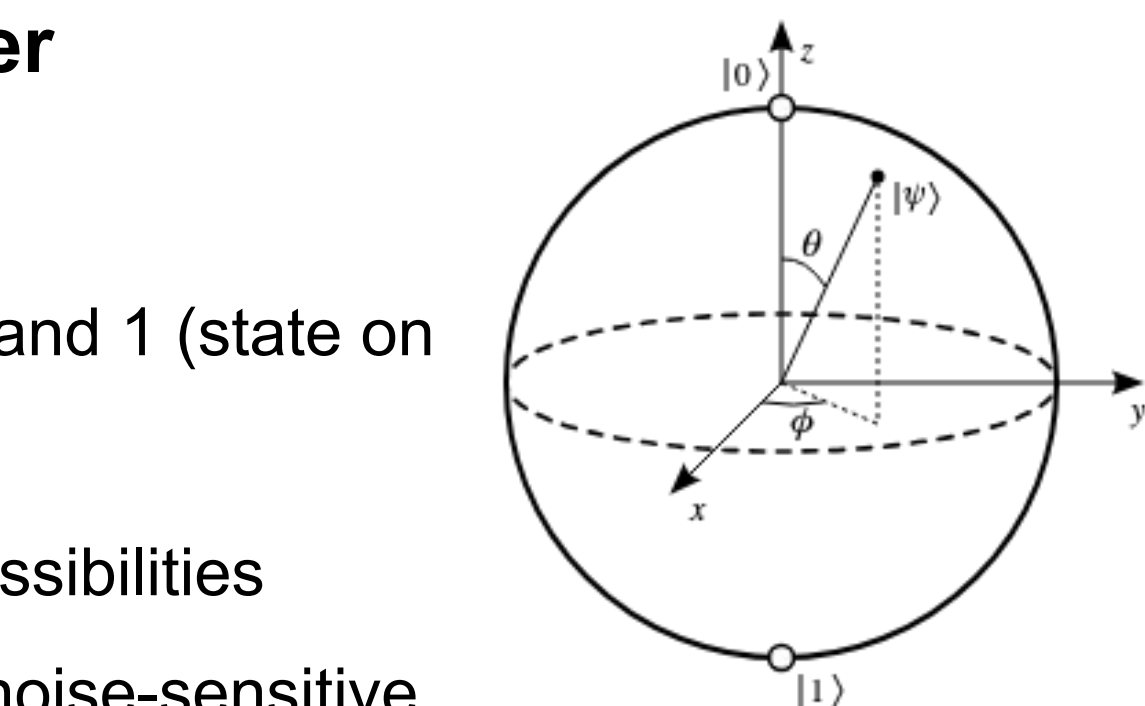
QML Real-World Applications

Medicine → Drug discovery

Chemistry → Molecular simulation

Biology → DNA analysis and modeling

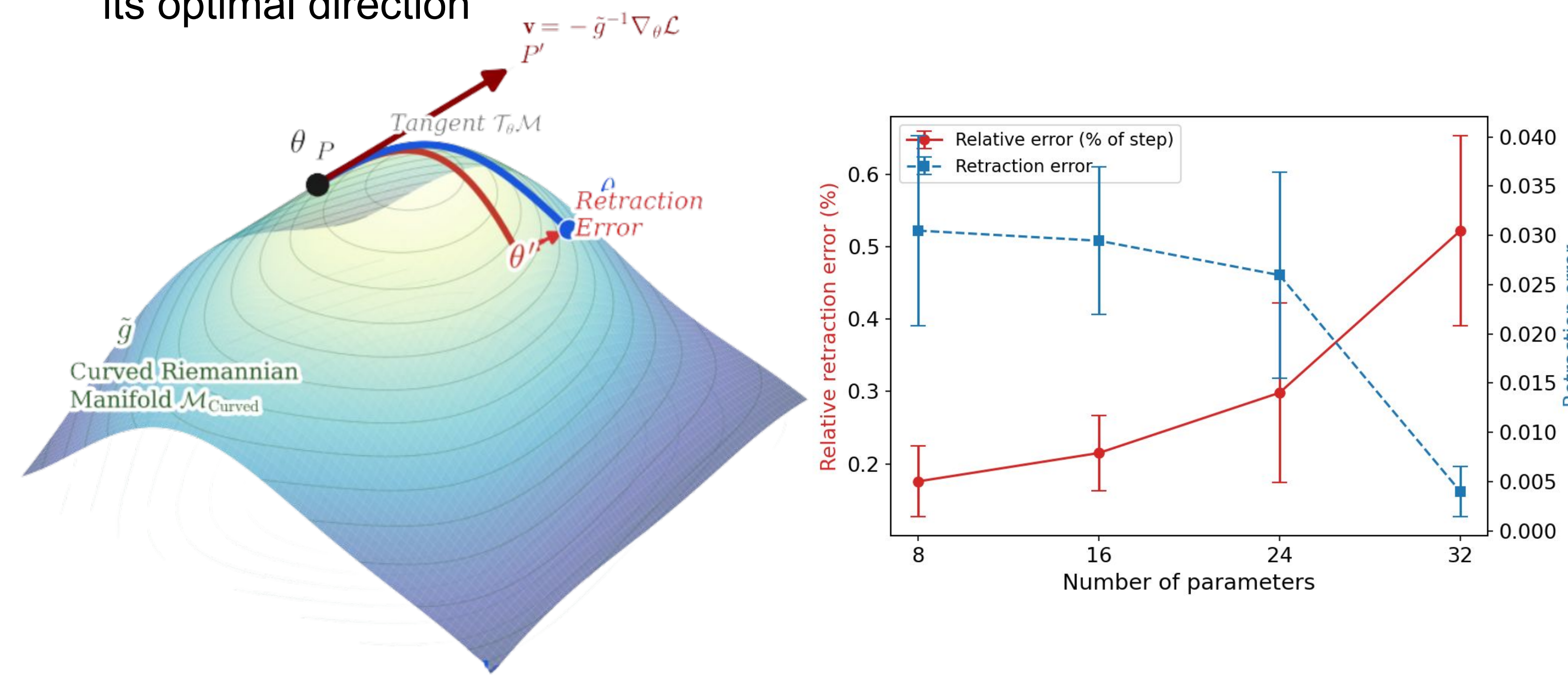
Finance → Trading optimization



The Flat Earth Paradox

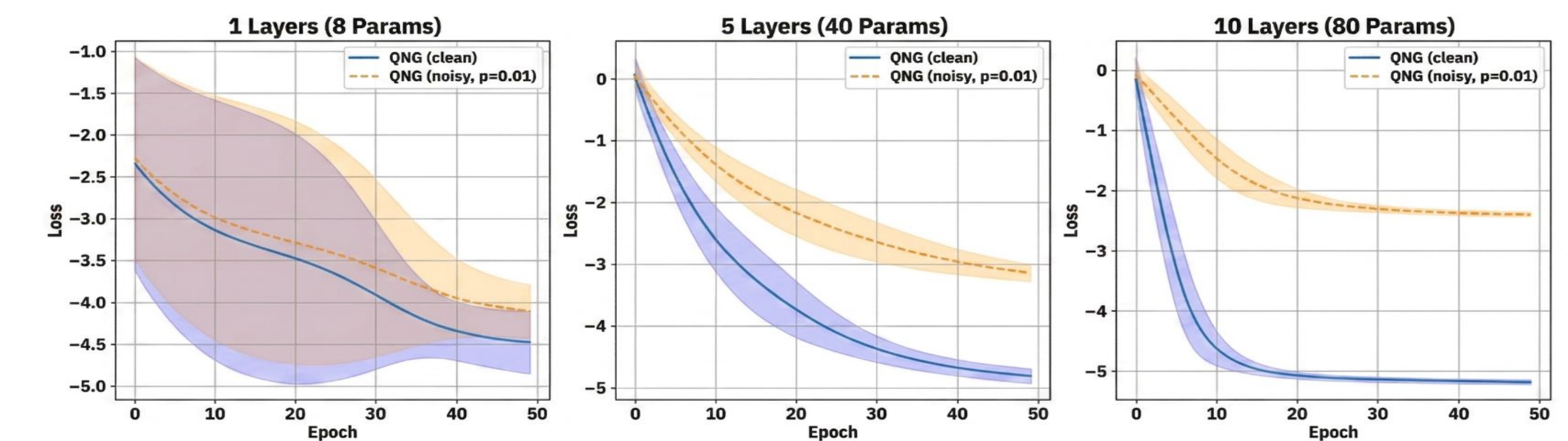
The fubini-study metric creates a curved quantum statistical manifold.

Quantum natural gradient maps the tangent vector through an Euler step on the ambient coordinate chart dropping the Levi-Civita connection of the manifold. This creates a retraction error forcing the descent path away from its optimal direction



Noise Problem

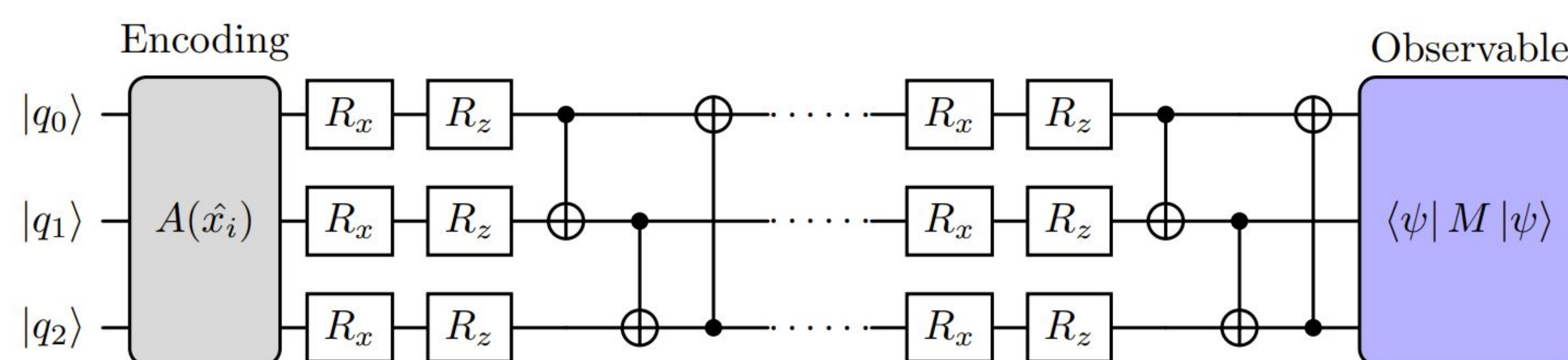
Theorem 3 (Noise-Induced Geodesic Deviation of QNG). Let $\mathcal{M}$ be the quantum statistical manifold with the regularized Fubini-Study metric $\tilde{g}$, subjected to local depolarizing noise with contraction parameter $q < 1$ and circuit depth L . While the Levi-Civita connection governing the manifold's intrinsic curvature remains invariant to leading order, the contravariant natural gradient vector field $\tilde{v}$ is exponentially amplified. Consequently, the geodesic deviation of the uncorrected ambient retraction diverges exponentially as $\|\tilde{\mathcal{E}}\|_{\tilde{g}} \in \mathcal{O}(q^{-2cL})$.



Solution: DFS-QNG

$$\mathcal{E}^k = (R_{\theta}(\eta v))^k - (\exp_{\theta}(\eta v))^k \approx \frac{\eta^2}{2} \sum_{i=1}^P \sum_{j=1}^P \Gamma_{ij}^k v^i v^j$$

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^P (\tilde{g}^{-1})_{kl} (\partial_i \tilde{g}_{jl} + \partial_j \tilde{g}_{il} - \partial_l \tilde{g}_{ij})$$



Algorithm 1 Dually-Flat Stratified Geodesic Quantum Natural Gradient (Exact QNG)

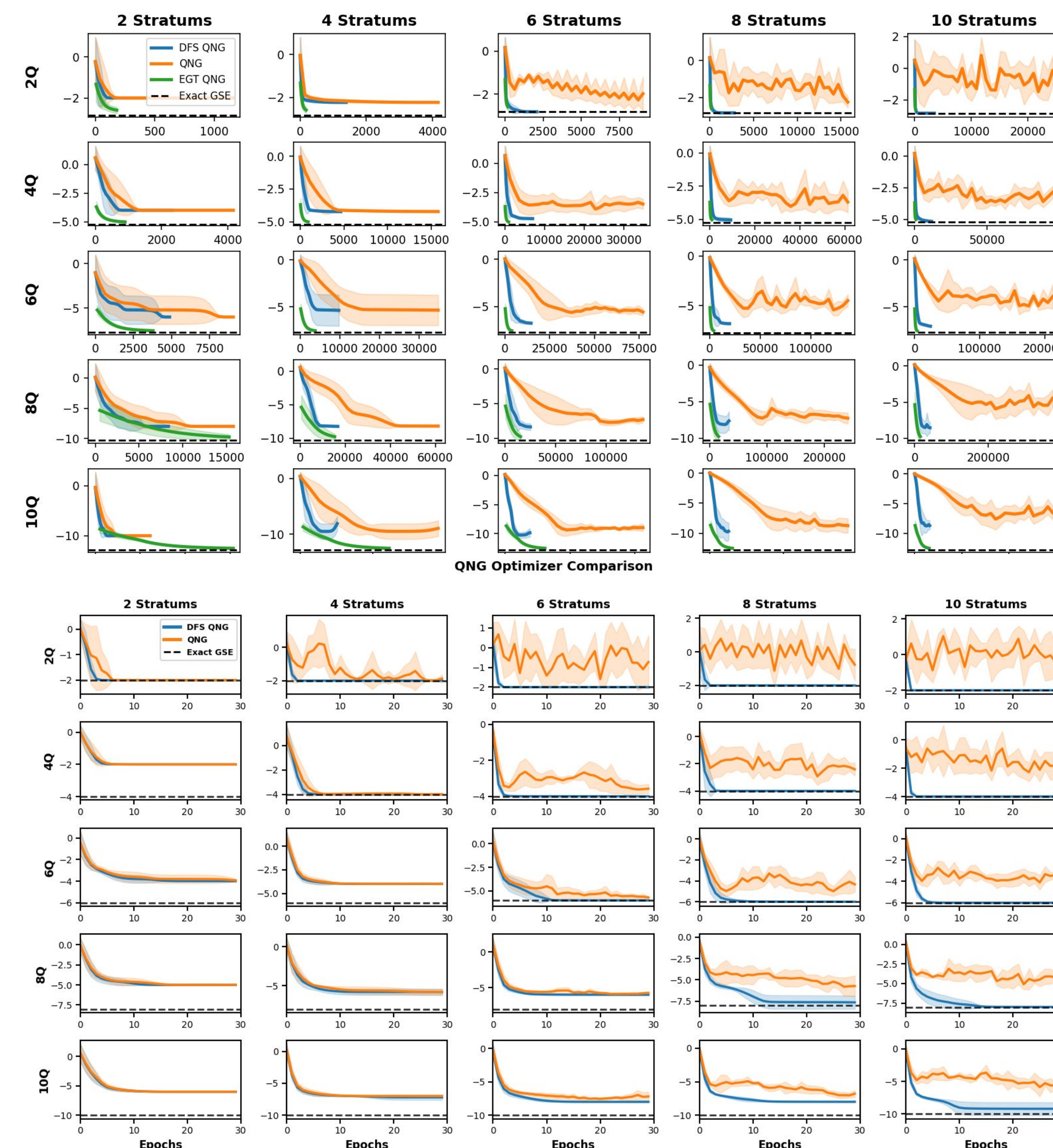
Require: Initial parameters $\theta^{(0)} \in \Theta \subseteq \mathbb{R}^P$; Abelian-stratified ansatz $\mathcal{U}(\theta)$ with stratification $f(g, \Theta) = \bigsqcup_{\pi=1}^S \mathcal{S}_{\pi}$ satisfying (16); shift parameter $s \in \mathbb{R}^+$; step size $\eta \in \mathbb{R}^+$; optimization horizon $T \in \mathbb{N}$.

- for $t = 0, 1, \dots, T-1$ do
- for $\pi = 1, 2, \dots, S$ do
- Submanifold Restriction.** Project onto $\mathcal{S}_{\pi}$: extract active coordinates $\theta_{\pi}^{(t)} \leftarrow \Pi_{\pi}(\theta^{(t)})$; freeze complement $\theta_{\pi}^{(t)} \leftarrow (I - \Pi_{\pi})(\theta^{(t)})$.
- Directional Derivative Evaluation.**
- for each $k \in \Theta_{\pi}$ do
- $\mathcal{L}_k^{\pm} \leftarrow \mathcal{L}(\theta^{(t)} \pm s e_k)$
- $[\nabla_{\pi} \mathcal{L}]_k \leftarrow (2 \sin s)^{-1} (\mathcal{L}_k^+ - \mathcal{L}_k^-)$
- end for
- Stratum Metric Evaluation.** Prepare $|\psi_{\pi}\rangle = \mathcal{U}_{\pi}(\theta_{\pi}^{(t)}) |\phi_{[1:\pi-1]}\rangle$ and evaluate:

$$\tilde{g}_{\pi\pi} \leftarrow \text{Re}[\langle \partial_i \psi_{\pi} | \partial_j \psi_{\pi} \rangle - \langle \partial_i \psi_{\pi} | \psi_{\pi} \rangle \langle \psi_{\pi} | \partial_j \psi_{\pi} \rangle]_{i,j \in \Theta_{\pi}}$$
- Geodesic Flow Integration.** Since $\Gamma_{ij}^{(e)k}|_{\mathcal{S}_{\pi}} = 0$, the exponential map reduces to coordinate translation:

$$\theta_{\pi}^{(t+1)} \leftarrow \theta_{\pi}^{(t)} - \eta \tilde{g}_{\pi\pi}^{-1} \nabla_{\pi} \mathcal{L}$$
- Chart Recombination.** $\theta^{(t+1)} \leftarrow \theta_{\pi}^{(t)} \oplus \theta_{\pi}^{(t+1)}$
- end for
- end for
- return $\theta^{(T)}$

Results: VQE & QAOA



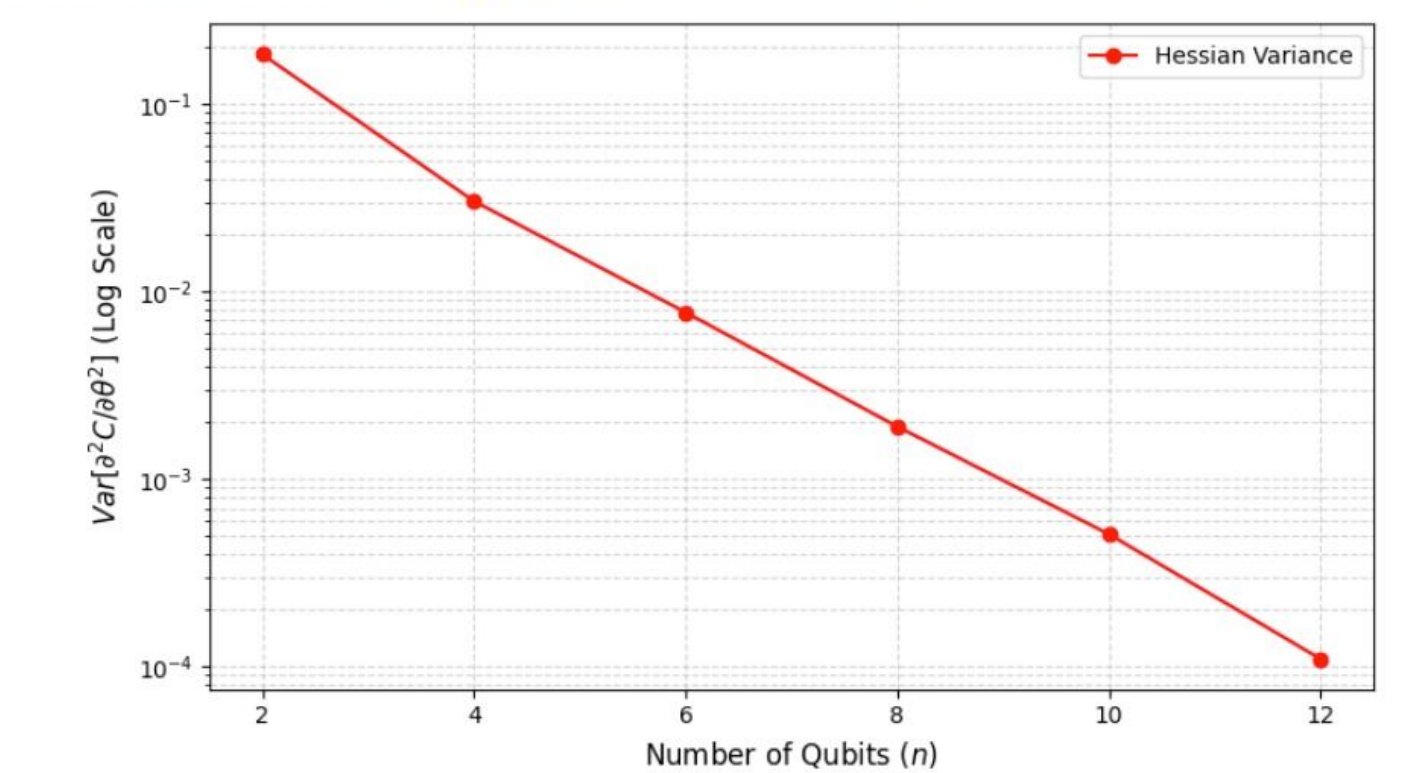
Geodesic Smoothness and Plateaus

Theorem 1 (Geodesic Smoothness). Let $\mathcal{M}$ be the quantum statistical manifold with the regularized Fubini-Study metric $\tilde{g} = g + \epsilon I$, and let $\mathcal{L} : \mathcal{M} \rightarrow \mathbb{R}$ be the objective function. Then $\mathcal{L}$ is geodesically L_R -smooth. Specifically, the metric-induced operator norm of the Riemannian Hessian, $\|\mathbf{H}^R\|_{\tilde{g}}$, is globally bounded by the constant L_R :

$$\|\mathbf{H}^R\|_{\tilde{g}} \leq L_R = \|\hat{\mathcal{O}}\|_2 / \epsilon \left[4 + 18/\epsilon \left(\sum_{k=1}^P \|\mathbf{g}_k\|_2^2 \right) \right] \left(\sum_{k=1}^P \|\mathbf{g}_k\|_2^2 \right) \Rightarrow L_R \sim \Theta\left(\frac{P^3}{\epsilon^2}\right) \quad (11)$$

Theorem 4 (Riemannian Barren Plateau). Let $\mathcal{M}$ be a quantum statistical manifold generated by a deep variational ansatz approximating at least a 2-design. The variance of the Riemannian Hessian elements is exponentially suppressed with respect to the number of qubits n , scaling as:

$$\text{Var}[\mathbf{H}_{ij}^R] \in \mathcal{O}\left(\frac{\text{Poly}(P)}{4^n}\right)$$



Acknowledgments

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